Int. J. Solids Structures Vol. 15, pp. 715–729
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0020–7683/79/0901–0715/$02.00/0

YIELD SURFACES AND LOADING SURFACES. EXPERIMENTS AND RECOMMENDATIONS†

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(*Received 22 June 1978; in revised form 27 December 1978*)

Abstract—The concepts of yield surface and loading surface at room temperature and elevated temperature are discussed and experimental results on pure aluminum in support of these concepts are presented. Recommendations for future experimental research are presented.

INTRODUCTION

During the last ten years a number of authors have proposed new theories of plasticity and viscoplasticity since it was argued that the so-called classical theories, that is, theories based on the existance of a yield surface and a plastic potential function, do not succeed to represent the experimental results correctly or that the classical theories are much too complicated to solve practical problems. Thus, for example, theories without a yield surface have been proposed. Unfortunately these nonclassical theories are even more complicated for practical use than the classical theories and in addition they do not seem to represent quantitatively the experimental results any better than the classical theories. The argument that the nonclassical theories represent a number of phenomena qualitatively is not necessarily an advantage over the classical theories, since also the latter in their general form, represent qualitatively the same phenomena well; in addition, some of the arguments concerning these same phenomena made in support of the nonclassical theories are occasionally based on dubious interpretations of available experimental results, for example, concerning the existance of cross effect. The above discussion should not be interpreted as an argument concerning the relative merits of the classical versus the nonclassical theories, but rather as a recognition that both classical and nonclassical theories can be useful and that their ultimate acceptability will depend on such improvements as will be needed to make them predict experimental results quantitatively well and to become sufficiently simple for practical use. This paper is devoted to the development of the classical theory.

During the last few years the senior author has shown that the classical theory should include, in addition to a yield surface, also a loading surface. The addition of the loading surface introduces a missing link which helps to make the classical theory agree better with experimental results, better understandable, and easier to use when applying it to practical problems. The concept of the loading surface was proposed first in [1], used in [2, 3] and shown to be valid experimentally in [4, 5].

The purpose of this paper is to present some new experimental evidence concerning yield surfaces, loading surfaces, and their interrelation. The experiments described in this paper were performed on commercially pure aluminum 1100–0, loaded in combined tension and torsion. We shall explore the relationships between the plastic strain increment vector, the loading path, the yield surface, and the loading surface, and on the basis of these experimental results we shall propose some recommendations for the development of a classical theory which satisfies the experimental evidence. After an introductory discussion of the concepts of yield surface and loading surface, we shall in the next section present the conclusions from these experiments and then in the following section we shall give our recommendations for future experimental research. Finally in the last two sections we shall present the experimental details and the description of the experiments.

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(a) Yield surfaces

The concept of the yield surface has more than one definition. It may mean the surface in stress space delimiting the region of purely linearly elastic behavior. It may also mean the surface in stress space delimiting a region which consists not only of the region of purely elastic behavior but also of the region in which some, agreed upon in advance, small amount of permanent strain upon loading is permitted. A different yield surface will be obtained in the same experiment according to the definition adopted and, in the second case, according to the amount of permanent strain accepted. It follows that there is some confusion in the published literature since different papers use different definitions of the yield surface. Thus, for example, the existence or nonexistance of cross effect depends on the definition adopted, and consequently the argument in support of a theory, that such theory shows or does not show cross effect is meaningless, unless the adopted definition of the yield surface is also taken into account; in other words, the theory should be able to generate yield surfaces which enclose an agreed upon in advance permanent strain region, and these yield surfaces should agree with the experimentally obtained yield surfaces based on the same definition.

In this paper we shall use the definition used consistently by A. Phillips *et al.*[6–8] which specifies that the yield surface delimits the region of purely elastic behavior and it is obtained by means of a well defined operational procedure. This procedure consists of making small excursions of the stress point into the plastic region, by amounts of plastic strain agreed upon in advance (usually 2 μ in/in but in some cases up to 5 μ in/in) and then backward extrapolation to the elastic line. For more information concerning this operational procedure we refer to the papers mentioned above.

In what follows we consider a six dimensional stress space and a six dimensional strain space. In the experiments, we deal with two dimensional subspaces of these six dimensional spaces.

In previous papers [7–9] we have shown experimentally that the yield surface moves in the direction of prestressing without any cross effect provided that the angle ϕ between the prestress vector $d\bar{\sigma}$ and the normal $\bar{n}$ to the yield surface at the origin of the prestress vector is small. Simultaneously, the yield surface changes its size in the direction of $d\bar{\sigma}$, and it becomes smaller when $d\bar{\sigma}$ is directed away from the origin, but it becomes larger when $d\bar{\sigma}$ is directed towards the origin. Thus, as suggested in [9, 10], at each point of the yield surface there must exist a neutral direction for $d\bar{\sigma}$ for which the yield surface will not change in size if prestressed in that direction. Since for this direction the yield surface will also show no cross effect, it follows that if prestressing occurs in the neutral direction, the yield surface will move essentially as a rigid body in the neutral direction. One of the tasks of the present experiments is to see whether the above conjecture is correct.

(b) Loading surface

In two recent papers [4, 5] we have shown that, in addition to the yield surface, there exists a loading surface which, for an initially isotropic material and for small strains only could be considered to be the Mises surface. The initial yield surface coincides with the initial loading surface but the subsequent yield surface is generally different from the corresponding subsequent loading surface. At any stage in the loading process the loading surface is the largest surface in stress space which for small strains is produced by isotropic expansion from the initial yield surface, and which passes through at least one previous prestress point. The essence of the concept of the loading surface is given in Fig. 1 and is well described in [5]. Let the stress path in stress space be given by the line OA. Then the loading surface I passes through A and is generated from the initial yield surface by means of isotropic expansion.

Suppose the stress point remains at A until all plastic and creep strains had time to develop; then the yield surface II, which is the boundary of the elastic region, will pass through the same point A, it will be tangential to I at A, it will be enclosed by I, and it will be much smaller than the loading surface through A. If the stress point does not remain at A until all plastic and creep strains had time to develop, then the yield surface II will not pass through A, but it will be enclosed by I; it will have such a form that a relatively small rigid body motion of this yield surface in the direction of A will make it pass through A and simultaneously to become tangential to I at A.

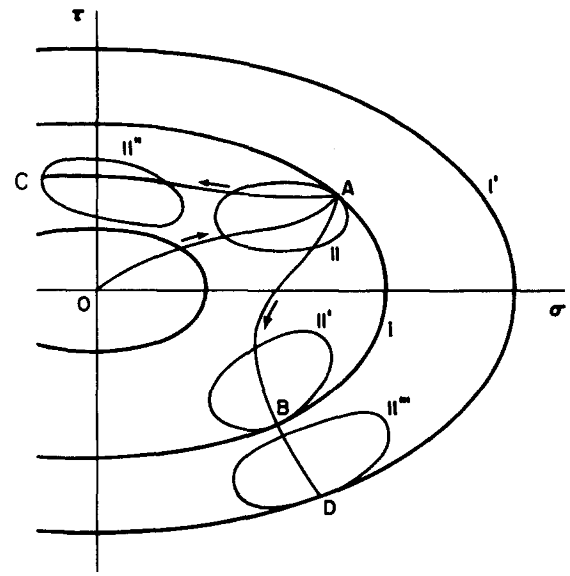

Fig. 1. Yield surfaces and loading surfaces.

Suppose we continue the stress path from A to B within I. Then, the loading surface I will remain unchanged while the yield surface will move and pass through B, if of course again all plastic and creep strains had time to develop while the stress point was stationary at B at the end of the path. The new yield surface II' will be tangential to I if B is on I or very near I. On the other hand, if the point B is sufficiently far from I, as is the case with the point C at the end of the path AC, then the new yield surface II'' which passes through C will be completely inside I and will not be tangent to I.

Suppose now that we continue our stress path from B to D (or from C to D) where D is outside I. Then the loading surface I changes to the new loading surface I' which passes through D. If again, while the stress point is stationary at D, all plastic and creep strains had time to develop, the new yield surface II''' will pass through D, will be tangential to I' at D and will lie completely inside I'. During the motion of the stress point which generates a new loading surface, $I \to I'$, the plastic and the creep strain rates generated are much larger than those strain rates generated by a motion of the stress point which keeps the loading surface unchanged.

The loading surface introduced above is essentially the six dimensional generalization of the prestressing point as long as the prestressing point is located outside any previous loading surface. We obviously assume the loading surface to be a Mises surface because of the isotropy of the stress space.

For the so-defined loading surface to have significance, it is necessary that the yield surface would not intersect it and that it will be essentially tangential to it if the stress point is located on it and we allow for sufficient time for the plastic and creep strains to develop. We indeed found this to be the case and consequently the so-defined loading surface is real. It should be added, however, that the loading surface should be thought more of the nature of a thin boundary layer in stress space than as a sharp delimiting line.

During prestressing, whether within the loading surface or when simultaneously moving it, plastic strains are generated and thus it is possible to obtain the direction of the plastic strain rate vector. An important question is whether the plastic strain rate vector is normal to the moving yield surface. In the present experiments we found that this is indeed the case. It follows then that since at the intersection between the stress path and the loading surface, the yield surface is tangential to the loading surface, the plastic strain rate vector must be normal to the loading surface. Our experiments show that this is also the case, and this finding gives an additional proof that the loading surface is real.

The definition of the loading surface as given here implies that it does not change when plastic strain develops while the yield surface moves inside the region delimited by the loading surface. We shall find that this implication is essentially valid.

The definition of the loading surface as given above is slightly different than the one given by Moon[11] which is based on the plastic strain rate magnitude. According to Moon, the loading surface is the surface for which, upon reloading, the plastic strain rate is equal to the last plastic strain rate obtained originally before unloading.†

(c) *The motion of the yield surface*

The experiments in [5] have shown that for angles ϕ, between the stress vector $d\bar{\sigma}$ and the normal $\bar{n}$ to the yield surface, which are not small, there is a motion of the yield surface not only in the direction of $d\bar{\sigma}$ but also in the direction of $\bar{n}$. The motion is therefore between the direction of $d\bar{\sigma}$ and that of the plastic strain increment vector $d\bar{\epsilon}^p$. We can state, therefore, that the motion of the yield surface is due to both $d\bar{\sigma}$ and $d\bar{\epsilon}^p$. For small ϕ the effect from either $d\bar{\epsilon}^p$ or $d\bar{\sigma}$ may be lost in the experimentation. The influence of $d\bar{\sigma}$ is a predominant one and we found that sometimes the influence of $d\bar{\epsilon}^p$ is completely missing. Whereas in specimen M-5 we obtained the yield surfaces at room temperature only, in specimens L-1 and L-2 we obtained yield surfaces for room temperature and for one elevated temperature for each prestressing. Thus, for specimens L-1 and L-2 it will be possible to identify a center for each yield surface pair by the method outlined in a previous paper[12]. This center is the "thermodynamic reference stress" defined earlier [13].‡ For specimens L-1 and L-2 we shall investigate the motion of the above defined center while for specimen M-5, where such a center cannot be defined, the yield surfaces are approximately elliptical in form and consequently we shall be able to investigate the motion of the center of these ellipses. Experiments M-5 and L-1 involve only very small plastic and creep strains, while experiment L-2 involves medium size strains (of the order of 1%) so that the results from these experiments can be considered valid for both small and medium size strains.

Experiments L-1 and L-2 will also allow us to investigate the existance of a loading surface at elevated temperature. We shall be able to conclude that a loading surface established by room temperature prestressing leads to a loading surface at elevated temperature. The relationship between the elevated temperature loading surface and the elevated temperature yield surfaces is the same as the relationship existing between loading surface and yield surfaces at room temperature.

CONCLUSIONS FROM THE EXPERIMENTS

The conclusions to be drawn from the experiments presented in this paper are:(1) the concept of the loading surface is valid for both room temperature and elevated temperature; the loading surface should be considered rather as a thin boundary layer than as a sharp demarcating line, (2) the position of the loading surface is independent of the amount of plastic and creep strain developed during the motion of the yield surface within the region enclosed by the loading surface, (3) the yield surface tends to be tangential to the loading surface whenever it is near it, (4) the plastic strain increment vector and the creep strain vector are normal to the yield surface, (5) as the stress path approaches the loading surface these two vectors tend to become normal to the loading surface which provides an additional proof of the existance of the loading surface, (6) when the stress path is along the Mises surface then the yield surface retains the same size in the direction of prestress and in the conjugate direction; when the stress path points away from the origin then the yield surface decreases in size in the direction of prestress but it shows no cross effect; when the stress path points towards the origin, the yield surface increases in size in the direction of prestress but it shows no cross effect, (7) the stress vector can be resolved in two components, one in the direction along the Mises surface and the other normal to this direction; each of the two components will affect the yield surface independently and the final yield surface will be the combination of these two effects, (8) the motion of the center of the yield surface is the result of the contribution of the prestress vector $d\bar{\sigma}$ and of the plastic strain increment vector $d\bar{\epsilon}^p$; however,

†This definition is also quite different from the concepts introduced by Sczepinski[17], Hecker[18], Dafalias and Popov[19], and Mroz[20].

‡This center is indeed the limiting center of the concentric set of yield surfaces at various temperatures. For specimens L-1 and L-2 the center was linearly extrapolated from data at two temperatures.

$d\bar{\sigma}$ predominates so that as a good approximation we can assume that the center of the yield surface moves in the direction of $d\bar{\sigma}$; this approximation is identical to the hardening law proposed by the senior author previously [8, 14] and the experimental results do not justify the use of any more elaborate hardening law which will take into account the direction of the plastic strain [15] or its modification [16], (9) when the stress point reaches the loading surface the yield surface becomes tangent to the loading surface, and if it is necessary for the yield surface to move sideways in order for the tangency to occur it will do so; it follows that it is not only the plastic strain but also the existance of the loading surface which makes the yield surface to move sideway and thus sometimes violate the conclusion 8; in particular, the influence of $d\bar{\epsilon}^p$ may be completely obliterated by the influence of the existance of the loading surface, (10) in all determinations of yield surfaces with pure aluminum no corner or pointed vertex was observed.

RECOMMENDATIONS FOR FUTURE EXPERIMENTAL RESEARCH

In this section we make a few recommendations concerning the development of a correct classical theory and point out questions which experimental research should answer in order for such a development to be facilitated. It seems to the authors that due to the existence of a loading surface distinct from the yield surface it is necessary to divide the stress path into those portions for which the loading surface increases in size and those portions which lie within the region enclosed by the existing loading surface. The fact that no such division is made in the nonclassical theories may be one of the reasons why such theories do not agree quantitatively well with experimental results. If the stress is stationary then appropriate creep strains must be added.

When the stress path is such that the loading surface increases in size the classical theory based on the Mises criterion and on normality is valid. When the loading surface is not increasing in size, that is, the stress path lies within the loading surface, we must first establish the yield surface. The exact form, size, and position of the yield surface is not important since the plastic strains generated while the path moves within the loading surface are very small, particularly in the region very near to the yield surface. However the approximate form, size, and position of the yield surface is important since in some problems the small strains generated while the stress point is located within the loading surface are of importance for their solution. The yield surface is always tangential to the loading surface if the stress point is on the loading surface, and as an approximation the yield surface can be considered to have an elliptical form; its size depends on its distance from the origin and it exhibits no cross effect. The law of change in size of the yield surface must be established experimentally; it is possible that enough experimental data already exist to express analytically this law for pure aluminum.

Once the path moves outside the yield surface but still remains inside the loading surface then plastic strains will be generated. The plastic strain increment vector will always be normal to the yield surface and the equivalent plastic strain rate will increase as we approach the loading surface. It can tentatively be assumed that this increase will be a function of the distance of the loading point from the loading surface in the direction of the stress increment vector. As the loading point is approaching the loading surface the plastic strain increases at an increasing rate so that when the stress point reaches the loading surface the tangent modulus of some equivalent stress-equivalent plastic strain curve is the same as before unloading has taken place. If the stress remains unchanged while it is within the loading surface but outside the yield surface, then creep strains will be generated and must be accounted for in the calculations.

It must experimentally be established: (1) how the plastic strain rate changes as the stress point moves from a yield surface to a loading surface on a path within the area enclosed by the loading surface, (2) what is the law of increase of the creep strains when the stress point is stationary within the loading surface or on it, (3) what is the law of increase of the plastic strain if the loading point is on the loading surface moving both the loading surface and the associated yield surface outwards. It is possible that sufficient experimental data already exist in order to answer one or more of these three questions.

EXPERIMENTAL DETAILS

The experiments described in this paper were performed on commercially pure aluminum

1100-0 with tubular specimens described in [6, 8]. The specimens were loaded in combined tension and torsion. The loading and unloading of the specimens was performed in a deadweight testing machine with the load being changed by small increments at the end of each of which the specimen strain was measured after the lapse of an agreed upon time interval. Hence, we had a deformational response of the specimen to the stress input. The preparation of the specimen, the deadweight testing machine, as well as the strain measuring equipment used are described in [5, 6, 8, 10].

Four strain gages were bonded to the outer surface of the specimens at middle length at locations 90° from one another. Two dummy gages were also used by mounting them on a piece of aluminum plate lightly glued to the specimen. In this way we had a completely temperature compensated system. The active gages were 45° rosette BLH-50-12s13 with orientation as shown in Fig. 2. The Wheatstone bridge circuits were as shown in Fig. 3.

In the $\sigma - \sqrt{(3\tau)}$ and $\epsilon^p - 1/\sqrt{(3)}\,\gamma^p$ spaces, the sensitivity for the strain measurements was $1/2\,\mu$in/in, the stress increments varied between 136 and 210 psi, and the stress rate varied between 14.8 and 76.3 psi/min, although in most prestressings, the stress rate ranged between 40 and 55 psi/min.

In these experiments only one specimen was used for the determination of the entire virgin yield surface and its subsequent yield surfaces. To obtain each indication of yield it is necessary to probe into the plastic region and therefore deform the yield surface while trying to determine it. It is necessary therefore to restrict each incursion into the plastic region to extremely small values, approximately 2–3 μin/in in plastic strain (equal to 4–6 times the readability of the instrumentation). The exact procedure of obtaining the proportional limit is explained in [6, 8].

One specimen was tested at room temperature only. The other two specimens were prestressed at room temperature, then the yield surface at room temperature was obtained, next the temperature was raised while the specimen was in the elastic range and the yield surface at the elevated temperature was obtained; then the temperature was decreased to room value and the next prestressing was initiated.

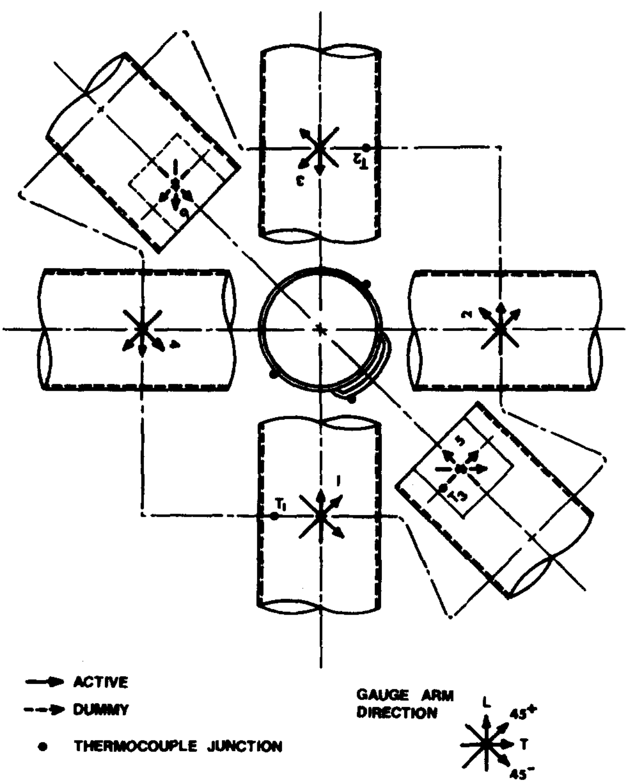

Fig. 2. Orientations of active gages.

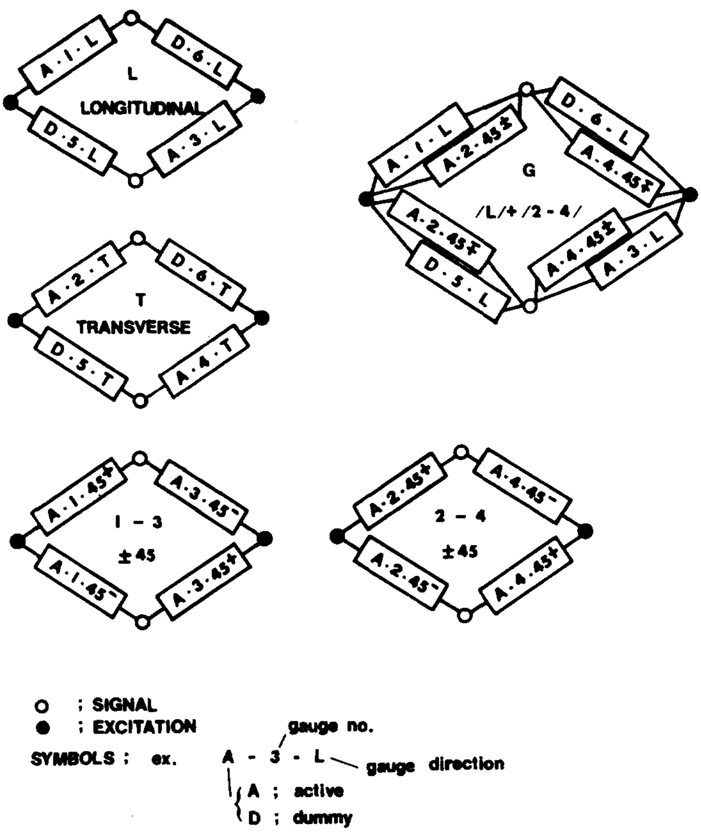

Fig. 3. Wheatstone bridge circuits.

DESCRIPTION OF EXPERIMENTS

Specimen M-5. This specimen was tested at room temperature. Seven subsequent yield surfaces *I–VII* and one loading surface were obtained originally with this specimen and are described in [5]. The sequence *VIII–XZ* of seven additional yield surfaces is part of the series of experiments described here. We shall, however, with the help of Fig. 4, describe briefly the prestressing history of this specimen by means of which yield surfaces *I–VII* were obtained. Prestressing in torsion was first used to obtain surface *I*. Then repeated prestressing from *A* to *B*, (*AB–CB–DB*), was used to obtain successively yield surfaces *II–IV*. Next repeated prestressing from *F* to *G*, (*FG–HG–KG*), was used to obtain successively the yield surfaces *V–VII*. The loading surface obtained during this process was the one passing through *B* and it was obtained at the prestressing *AB*. The purpose of obtaining yield surfaces *I–VII* was different from the one we proceed to describe next and which is central to this paper.

After nearly complete unloading ($\sigma = 175$ psi, $\tau = 0$) we obtained yield surface *VIII*, Fig. 5†, and then prestressed at constant rate radially from $M(\sigma = 2176$ psi, $(\sqrt{3})\tau = -1283$ psi) to $N(\sigma = 5197$ psi, $(\sqrt{3})\tau = -3064$ psi). During this prestressing we crossed the previously established loading surface passing through the point *B*, the intersection of which with the path *MN* is the point $M'(\sigma = 4479$ psi, $(\sqrt{3})\tau = -2641$ psi). During the prestressing *MN* the plastic strain increment vector was originally normal to the yield surface *VIII* and gradually rotated so that at M' it became nearly normal to the loading surface through M' and it retained nearly the same direction until the point *N* was reached. The amount of plastic strain developed during prestress *MN* was very small, $\Delta\epsilon^p = 10\,\mu$in/in, $\Delta\gamma^p/\sqrt{(3)} = -10\,\mu$in/in. The slope of the $\sqrt{(3)}\epsilon^p/\gamma^p$ curve was 0 at *M* and gradually changed to 1.3 at M' and at *N*. The slope of the curve $\sigma/\sqrt{(3)}\tau$ is of course 1.7 which is different than 1.3 but the difference is within the possibility of experimental error due to the small value of the strains. After arrival at *N* the specimen was kept under constant stress for 60 hr while creep strain developed which

†Remark that the coordinates in Fig. 5 are different from those in Fig. 4.

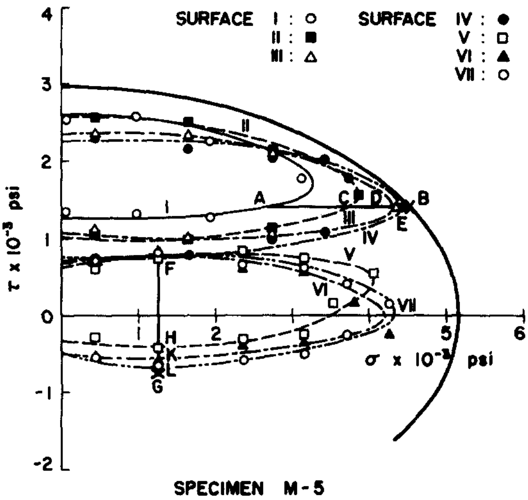

Fig. 4. Specimen M-5. Yield surfaces I–VII.

finally subsided. The direction of the creep strain vector oscillated slightly but the amplitude of this oscillation was small and the average direction was the normal to the loading surface at N. Indeed the total creep strains at N were $\Delta\epsilon^c = 11.5\,\mu$in/in and $\Delta\gamma^c/\sqrt{(3)} = -7\,\mu$in/in, with a ratio 1.60, which is quite near to 1.70 for the stresses.

By reaching N, a new loading surface was established and the new yield surface IX is tangential to the loading surface at N. The history of the change in direction of the plastic strain increment vector outlined above shows that the concept of the loading surface is a genuine one; indeed the loading surface was established at B but it became also effective a M'. In addition, the yield surface at N is tangential to the loading surface at N.

We observe that the size of the yield surface IX in the direction MN is smaller than that of surface $VIII$ in the same direction, while the size of IX in the conjugate to MN direction is the

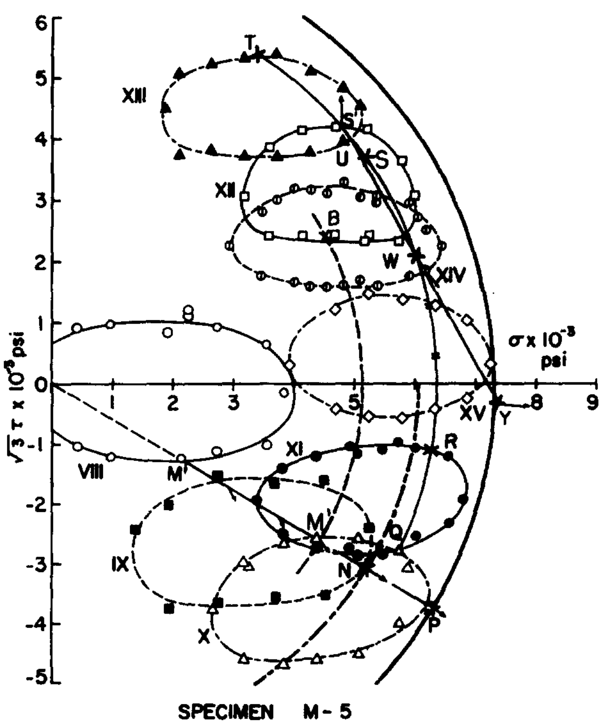

Fig. 5. Specimen M-5. Yield surfaces VIII–XIV.

same as that of *VIII* in the same conjugate direction. This result is in agreement with our conjecture that the yield surface decreases in size in the direction $d\bar{\sigma}$ but there is no cross effect (in the conjugate direction). This phenomenon was also seen in our previous experiments [3].

If the yield surfaces could be assumed approximated by ellipses and the centers of these ellipses considered to be the centers of the yield surfaces we would conclude that the motion of the center of *VIII* to the center of *IX* is the resultant of a contribution from $d\bar{\sigma}$ and a contribution from $d\bar{\epsilon}^p$. However, during prestressing the yield surface must move sideways in order to become tangential to the loading surface at N. Hence, the motion of the yield surface could also be considered as due to the contribution from $d\bar{\sigma}$ alone and to the existance of the loading surface, without the need to consider a contribution from $d\bar{\epsilon}^p$.

The next prestressing NP is radial and it establishes a new loading surface through $P(\sigma = 6286\,\text{psi},\ \sqrt{(3)}\tau = -3709\,\text{psi})$ to which the new yield surface X is tangential. We again observe that the size of the surface X in the direction of prestressing is smaller than that of surface *IX*, which in the conjugate direction the sizes of the two surfaces are the same. During prestressing NP a more substantial increase in the plastic and creep strains occurs (than for MN). The total plastic strains during the path NP are $\Delta\epsilon^p = 103\,\mu\text{in/in}$ and $\Delta\gamma^p/\sqrt{(3)} = -78\,\mu\text{in/in}$. After arrival at P the specimen was kept at constant stress for 50 hr while creep strains developed. The total creep strains at P were $\Delta\epsilon^c = 75\,\mu\text{in/in}$, $\Delta\gamma^c/\sqrt{(3)} = -47\,\mu\text{in/in}$. The direction of the plastic strain increment vector at P has the ratio 1.33, while that of the creep strain vector has the ratio 1.61. These two ratios should be compared with the ratio 1.70 for the normal to the loading surface. The motion of the center of the yield surface is in the direction of $d\bar{\sigma}$.

The next prestressing QR is in the direction tangential to the Mises surface at Q. This prestressing was accomplished in three steps which divided equally the total arc length. At the end of each of these steps there was a 13.5, 5 and 14 hr stay respectively. This step-by-step prestressing was taken in order to minimize the effect of the final stress value, which might otherwise be great, so that the subsequent yield surface could follow continuously the exact prestressing path. We observe that the yield surface moves in the neutral direction and remains normal to the path. We observe that the size of the yield surface in both the prestressing and the conjugate directions remains unchanged as our conjecture predicted. The motion of the center of the yield surface is approximately in the neutral direction. The amount of plastic strain and creep generated was too small to make it possible to obtain plastic strain increment directions and creep strain directions.

The next prestressing RS led to yield surface *XII*. This prestressing was also in the tangential direction to the Mises surface through R, and it was accomplished in four steps at the end of each of which there was a stay of 18, 23, 23 and 48 hr respectively. The end point $S(\sigma = 5192\,\text{psi},\ \sqrt{(3)}\tau = 3709\,\text{psi})$ is now within the yield surface *XII* which implies that during the 48 hr stay at S the yield surface overshoot the point S. This phenomenon was observed also in [10]. The sizes of *XII* in the direction of prestressing and conjugate to it are approximately the same as those of *XI* in those few directions which shows again our conjecture to be correct. The motion of the center of the yield surface is also approximately in the neutral direction. The total amount of plastic and creep strain developed during the prestressing QS was $14\,\mu\text{in/in}$ in the ϵ direction and 32 in/in in the $\gamma/\sqrt{(3)}$ direction. Most of the plastic strain and creep strain developed during the last leg of the loading RS. For this portion it became possible to obtain the direction of the plastic strain increment vector and of the creep strain vector. These directions are approximately normal to the yield surface at S'.

The next prestressing ST to the point $T(\sigma = 3393\,\text{psi},\ \sqrt{(3)}\tau = 5402\,\text{psi})$ was again in the tangential direction to the Mises surface through S and led to surface *XIII* which was obtained without any stay at T but nevertheless the surface passed through T. The total amount of plastic strain developed during the prestressing ST was $\Delta\epsilon^p = 2\,\mu\text{in/in}$, $\Delta\gamma^p/\sqrt{(3)} = 22\,\mu\text{in/in}$. The plastic strain increment vector is normal to the yield surface at T. It has the same direction throughout the motion ST. We observe that the motion ST of the yield surface is in the direction of $d\bar{\sigma}$. The sizes of surface *XIII* in the prestressing and conjugate directions are again approximately the same as for surface *XII*. The center of the yield surface moves in the direction of prestressing.

The next prestressing UW leads to surface XIV while the last prestressing WY leads to surface XV. Point $W(\sigma = 6032\,\text{psi}, \sqrt{(3)}\tau = 2096\,\text{psi})$ is within surface XIV after a stay of 22 hr at W, while point $Y(\sigma = 7393\,\text{psi}, \sqrt{(3)}\tau = -327\,\text{psi})$ is outside surface XV with a stay of only 1 hr at Y. The point Y is on the loading surface and we observe that XV is tangential to the loading surface not at Y but at another neighboring point. The plastic strain increment vectors start by being normal to the yield surfaces $XIII$ and XIV but gradually rotate in the direction of increasing $\Delta\epsilon^p$. The creep strain rates accentuate this rotation which may be due to the fact that the path UWY approaches the loading surface and consequently the plastic strain vector tries to become normal to the loading surface. The plastic strain increment vector and the creep strain vector at Y is normal to the loading surface at Y. The center of the yield surface moves in the direction of $d\bar{\sigma}$ and not appreciably in the direction of $d\bar{\epsilon}^p$ when this last direction is not coincident with that of $d\bar{\sigma}$. The total amount of plastic and creep strain during the prestressing UY is $\Delta\epsilon^{p+c} = 23\,\mu\text{in/in}$, and $\Delta\gamma^{p+c}/\sqrt{(3)} = -10\,\mu\text{in/in}$.

As a final remark concerning this test M-5 we can say that the amount of plastic strain and creep strain generated while the yield surface was moving within the loading surface did not affect the position of the loading surface. Indeed, in this test the total amount of plastic and creep strain generated during the path $QRSTUWY$ was $\Delta\epsilon^{p+c} = 39\,\mu\text{in/in}$, $\Delta\gamma^p/\sqrt{(3)} = 64\,\mu\text{in/in}$. The same conclusion can be drawn by considering the experiments by Phillips and Moon[5]. For example in experiment M-4 in[5] the prestressings 5–9 did not affect the loading surface despite the fact that during these prestressings a total amount of plastic and creep strain equal to $\Delta\epsilon^{p+c} = 21\,\mu\text{in/in}$; $\Delta\gamma^{p+c}/\sqrt{(3)} = 61\,\mu\text{in/in}$ was generated.

Specimen L-1. Specimen L-1 was used to obtain the initial yield surface, four subsequent yield surfaces I–IV and one loading surface. Although all prestressings were at room temperature, the initial and subsequent yield surfaces were obtained by means of yield curves at room temperature and at 190°F. Thus, the centers of these yield surfaces could be determined by a method outlined in[12]. These centers may have significance from the thermodynamic point of view as shown in[12].

From Fig. 6 we observe that the initial yield surface is not isotropic. The reason for this anisotropy is that, while attempting to obtain the initial yield surface, we accidentally obtained excessive plastic strain so that in reality what is supposed to be the virgin yield surface is a prestressed yield surface.

The first deliberate prestressing was a radial one from A to B $(\sigma = 5576\,\text{psi}, \sqrt{(3)}\tau = 4836\,\text{psi})$. This prestressing generated the only loading surface obtained with this specimen. The specimen remained at B for only 0.5 hr so that the yield surface I does not pass through B. The form of the yield surface is such that it can be assumed that had the specimen remained at B for longer time, the yield surface would have become tangential to the loading surface at B. The amount of plastic strain developed during prestressing AB was very small, possibly because of the overloading mentioned above.

The second prestressing was partly radial and partly along the Mises surface: $CDCEFEGHGJKJLM$. The prestressing consists therefore of short radial motions to the loading surface at D, F, H, K and M and return to the starting points C, E, G, J and L and of short motions along the Mises surface CE, EG, GJ, JL. Every time the specimen reached the loading surface at D, F, H, K, M, it remained there for 0.5 hr before retreating to the starting point.

We observe that the motion of the yield surface is in the direction along the Mises surface. In fact the centers 0_I and 0_{II} of the surfaces I and II lie on the same Mises surface. This motion is similar to the motion we observed in experiment M-5. It is interesting that the short prestressing to the loading surface did not change the outcome. The total amount of plastic strain developed during this prestressing was very small, $\Delta\epsilon^p = 12\,\mu\text{in/in}$, $\Delta\gamma^p\sqrt{(3)} = 1\,\mu\text{in/in}$.

A comparison of surface II with surface I reveals that surface II has approximately the same size as surface I in both the direction tangential to the Mises surface and in the conjugate direction. This result is similar to the one obtained with the previous experiment. Obviously a motion of the yield surface in the direction along the Mises surface does not change the size of the yield surface.

The two surfaces I and II are shown also in Fig. 7. We observe that the yield curves at room temperature are tangential to the same Mises curve $B'M'$. Also we observe that the two yield

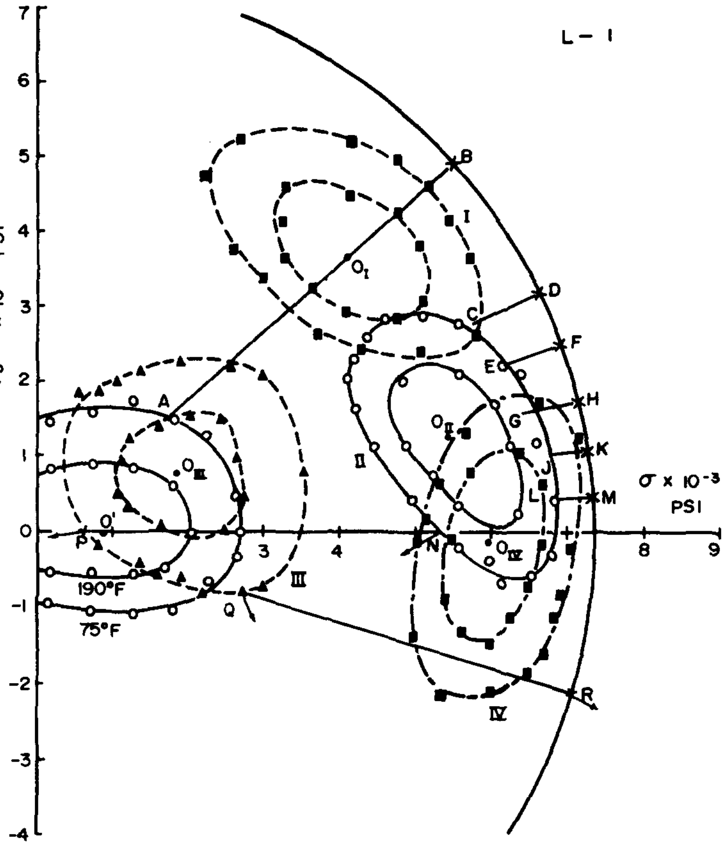

Fig. 6. Specimen *L*-1. Yield surfaces *I–IV*.

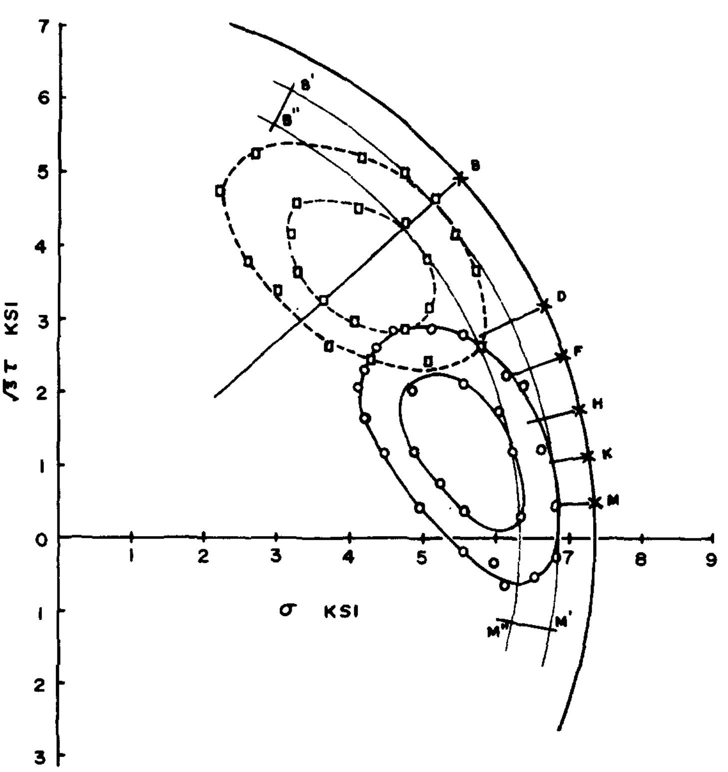

Fig. 7. Specimen *L*-1. Loading surfaces at elevated temperatures.

curves at 190°F are tangential to another Mises curve $B''M''$. This observation implies that the concept of the loading surface is valid not only for room temperature but also for elevated temperature. In other words, once a loading surface is established at room temperaure, there exists a family of loading surfaces for all temperatures. If at room temperature the yield surface is tangential to the loading surface at this temperature, then at an elevated temperature, the elevated temperature yield surface will be tangent to the corresponding elevated temperature loading surface.

We return to Fig. 6 and observe that the next prestressing was NP producing nearly complete unloading. The stay at the point P was again 0.5 hr. The plastic strain increment vector at N is normal to yield surface II and that at P it is nearly normal to yield surface III. The size of the yield surface III in the direction $d\bar{\sigma}$ is larger than that of surface II and in the conjugate direction the sizes of these two surfaces are approximately equal (no cross effect). This observation is in accordance with previous observations[3–5] that a prestressing towards the origin increases the size of the yield surface in the direction of prestressing. The motion of the center of the yield surface reveals an influence of both $d\bar{\sigma}$ and $d\bar{\epsilon}^p$.

The last prestressing was QR in the radial direction and the resulting yield surface was IV with center O_{IV}. The stay at R was again 0.5 hr. Surface IV is smaller in size than III in the prestressing direction, as it should, but it also shows some cross effect. In addition the motion of the center shows the influence of $d\bar{\sigma}$. The plastic strain increment vector at Q is normal to surface III and at R it is nearly normal to the loading surface. This result is as expected.

Specimen L-2. This specimen was used to obtain seven subsequent yield surfaces and six loading surfaces. All prestressings were at room temperature but the yield surfaces were obtained at room temperature and at 190°F. Thus, the centers of the yield surfaces could be determined by the method outlined in[12]. In contrast to the previous two specimens, specimen L-2 showed much larger plastic and creep strains, particularly beginning with the third prestressing. Thus, it was possible to see whether some of the previously obtained results were valid for larger plastic and creep strains. In Fig. 8 we see the initial yield surface which is then

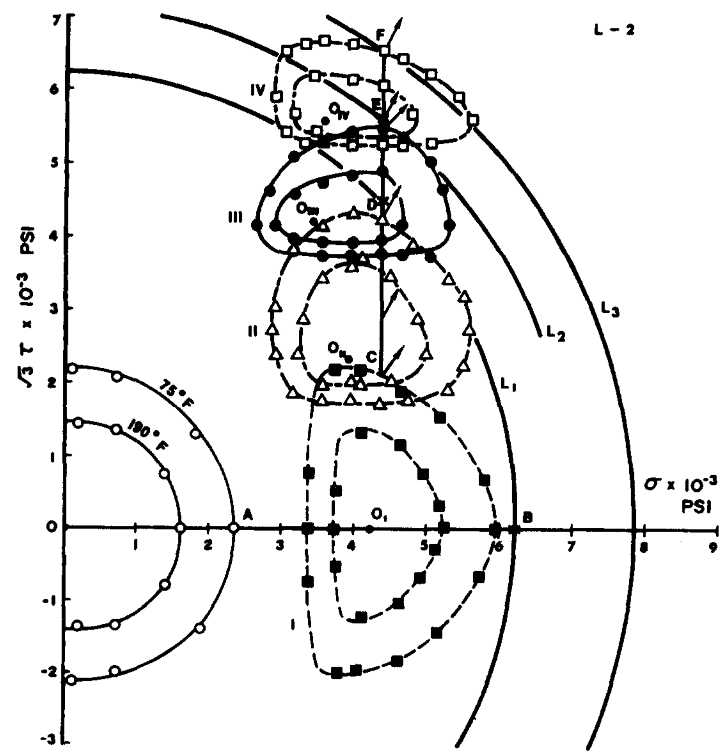

Fig. 8. Specimen L-2. Yield surfaces I–IV.

prestressed radially AB thus generating a loading surface L_1 through B and a yield surface I with center O_I. The stay at B was 20 hr and the yield surface passes near B. The plastic strain increment vector is normal to the yield and loading surface. The yield surface is tangential to the loading surface, it shows no cross effect and its size in the direction of prestressing decreases as it should since the motion is away from the origin. Point B has the coordinates ($\sigma = 6245$ psi, $\sqrt{(3)}\tau = 0$) and the plastic strain developed during prestressing AB was $\Delta\epsilon^p = 15\ \mu$in/in while the creep strain at B was $\Delta\epsilon^c = 15.5\ \mu$in/in.

The next prestressing CD to the point D on L_1($\sigma = 4361$ psi, $\sqrt{(3)}\tau = 4466$ psi) generated a new yield surface II with the center O_{II}. This yield surface is nearly tangential to the loading surface L_1 but not at D. The surface does not pass through D although the stay at D was 60 hr. The total plastic strain developed during prestressing CD was $\Delta\epsilon^p = 3.5\ \mu$in/in, $\Delta\gamma^p/\sqrt{(3)} = 6\ \mu$in/in, while the creep strain was $\Delta\epsilon^c = -3.5\ \mu$in/in, $\Delta\gamma^c/\sqrt{(3)} = 10.5\ \mu$in/in. The negative value in the creep strain was due to a gradual change in the direction of the creep strain vector. The plastic strain increment vector is normal to surface I at C and to surface II at D. The creep strain vector was first erratic in its direction but finally became normal to the yield surface at D. This direction is little different from the normal to the loading surface.

The size of the yield surface II in the prestressing direction is smaller than that of yield surface I since we are moving away from the origin and there is no cross effect. The center O_{II} is approximately over and to the left of O_I which shows the influence of $d\bar{\sigma}$ and of the need for the yield surface to be tangent to the loading surface. Observe also that the influence of $d\bar{\epsilon}^p$ would have required the yield surface to move towards the right instead of towards the left.

The next prestressing DE to the point E($\sigma = 4361$ psi, $\sqrt{(3)} = 5583$ psi) generates a new loading surface L_2 and a new yield surface III tangential to L_2 near E; the stay at E was 82 hr. This prestressing generated a relatively large amount of plastic strains, $\Delta\epsilon^p = 272\ \mu$in/in, $\Delta\gamma^p/\sqrt{(3)} = 172\ \mu$in/in, while the creep strain was $\Delta\epsilon^c = 4\ \mu$in/in, $\Delta\gamma^c/\sqrt{(3)} = 66\ \mu$in/in. The plastic strain increment vector rotated sufficiently be become normal to L_2 at E. The size of the yield surface III in the prestressing direction is smaller than that of surface II and there is no cross effect. The center O_{III} is over and to the left of O_{II} showing again the influence of $d\bar{\sigma}$ and of the need for the yield surface to be tangent to the loading surface. Again remark that the influence of $d\bar{\epsilon}^p$ would have required the yield surface to move toward the right instead of towards the left.

The next prestressing EF generates a new yield surface IV and a new loading surface L_3. Surface IV is nearly tangential to L_3. The stay at F($\sigma = 4361$ psi, $\sqrt{(3)}\tau = 6540$ psi) was 104 hr. A large amount of plastic strain was developed during prestressing: $\Delta\epsilon^p = 1110\ \mu$in/in, $\Delta\gamma^p/\sqrt{(3)} = 2300\ \mu$in/in. The amount of creep generated at F was more modest: $\Delta\epsilon^c = 80\ \mu$in/in, $\Delta\gamma^c/\sqrt{(3)} = 500\ \mu$in/in. The plastic strain increment vector was finally normal to IV and the creep strain vector was normal to L_3. We again observe no cross effect and a decrease in the size of the yield surface which is expected since we moved away from the origin. The center O_{IV} is above O_{III} and towards the right showing again the influence of $d\bar{\sigma}$ and possibly of $d\bar{\epsilon}^p$.

The next prestressing FG (Fig. 9), generates the yield surface V which is tangential to the new loading surface L_4. The stay at G($\sigma = 5842$ psi, $\sqrt{(3)}\tau = 6540$ psi) was 122 hr. The amount of plastic strain developed was large: $\Delta\epsilon^p = 1220\ \mu$in/in, $\Delta\gamma^p/\sqrt{(3)} = 1480\ \mu$in/in. The amount of creep generated at G was modest: $\Delta\epsilon^c = 60\ \mu$in/in, $\Delta\gamma^c/\sqrt{(3)} = 120\ \mu$in/in. The plastic strain increment vector and the creep strain vector were normal to the yield and loading surfaces at G. We observe no cross effect and a decrease of the size of the yield surface in the direction of prestressing. The position O_V relative to O_{IV} shows the influence of $d\bar{\sigma}$ and the relative lack of influence of $d\bar{\epsilon}^p$.

Prestressing GK is in the direction tangential to the Mises surface and the yield surface at G. It generates a yield surface VI which is nearly tangential to the new loading surface L_5. The stay at K($\sigma = 8534$ psi, $\sqrt{(3)}\tau = 4148$ psi) was 70 hr. The amount of plastic strain developed was smaller than in the preceeding two prestressings: $\Delta\epsilon^p = 550\ \mu$in/in, $\Delta\gamma^p/\sqrt{(3)} = 300\ \mu$in/in. The reason for these smaller values is, possibly, that the loading was in a direction very near to a neutral direction for both the yield surface and the loading surface. Indeed the insert in the figure shows that only after the position R, the plastic strain started increasing appreciably. The amount of creep generated at K was $\Delta\epsilon^c = 125\ \mu$in/in, $\Delta\gamma^c/\sqrt{(3)} = 50\ \mu$in/in. The plastic strain

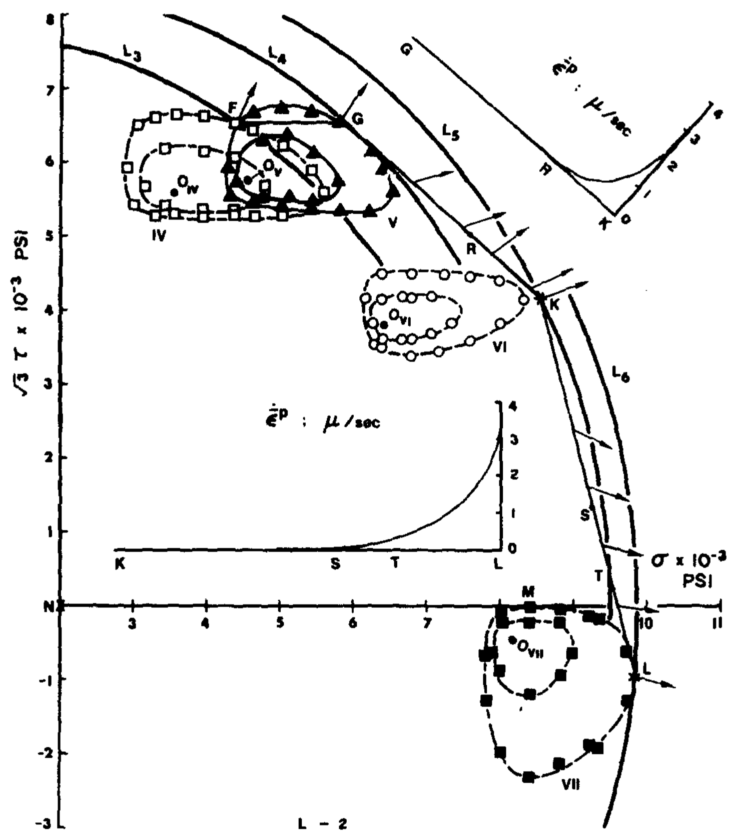

Fig. 9. Specimen L-2. Yield surfaces V–VII.

increment vector and the creep strain vector are nearly normal to the yield surface and loading surface at K. We observe that the yield surface moves nearly in the direction of prestressing, that there is no cross effect and that the size of the yield surface in the prestressing direction becomes smaller. We also observe that the direction of the line $O_V O_{VI}$ compared to the direction GK reveals an influence of $d\bar{\sigma}$ but not of $d\bar{\epsilon}^p$. Indeed if $d\bar{\epsilon}^p$ were influential it would have required O_{VI} to be much more near L_5 than it really is.

The last prestressing KL was selected such that it will intersect the loading surface L_5 but will generate a new loading surface L_6 at only a short distance from L_5. The new yield surface VII was tangential to L_6 at L. The stay at $L(\sigma = 9826\,\text{psi},\ \sqrt{(3)}\tau = -957\,\text{psi})$ was 80 hr. The amount of plastic strain generated was considerable: $\Delta\epsilon^p = 2370\,\mu\text{in/in}$, $\Delta\gamma^p/\sqrt{(3)} = -445\,\mu\text{in/in}$. The creep strain generated at L was $\Delta\epsilon^c = 160\,\mu\text{in/in}$, $\Delta\gamma^c/\sqrt{(3)} = -55\,\mu\text{in/in}$. For the first half KS of the path between K and L the plastic strains are very small and the direction of the plastic strain increment vector gradually rotates from the one shown at K to those directions shown near S. Between S and L the direction of the plastic strain increment vector as well as that of the creep strain vector are constant and nearly normal to L_6. The insert in the figure shows that after the position T at the intersection between the path and L_5, the plastic strain started increasing appreciably. The yield surface moves in the direction of prestressing and the direction of line $O_{VI}O_{VII}$ shows the influence of $d\bar{\sigma}$ and the need for the yield surface to be tangent to the loading surface and possibly of $d\bar{\epsilon}^p$. However, the influence of $d\bar{\sigma}$ predominates as in all previous cases. There is no cross effect, but the width of the yield surface in the direction of prestressing is now much larger than before despite the fact that the yield surface moves slightly away from the origin. This is a case where our general rule is not valid.

Three remarks should now be made. First, that the yield surface sometimes intersects the loading surface slightly and sometimes does not reach it by a small amount despite a long stay at the prestressing point. Therefore, we should consider the loading surface more in the nature of a boundary layer in the stress space than as a sharp delimiting line. The second remark is that the total amount of plastic and creep strain developed during test L-2 was $\epsilon^{p+c} = 5988\ \mu\text{in/in}$, $\gamma^{p+c}/\sqrt{(3)} = 5504\ \mu\text{in/in}$. Thus, the concept of the loading surface is valid not only for very small plastic strains but also for plastic strains of the order of 1%. The last remark is that in these experiments no evidence of a corner in any yield surface was observed.

Acknowledgements—The authors are indebted to Mr. W. Kawahara and W. Y. Lu, graduate students at Yale University, for their help in this research. They also gratefully acknowledge the helpful criticisms of Prof. M. A. Eisenberg of the University of Florida who read an earlier version of this paper. The support of this research by the National Science Foundation is gratefully acknowledged.

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